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Critical exponents of the n-component model via renormalisation group recursion formulae for dimension between 2 and 4

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Abstract. We study the n-component φ^4 model in space dimensions $2 \leqslant d \leqslant 4$, for various values of n using the Wilson recursion formula. In two dimensions, asymptotic freedom is seen when $n \geqslant 3$, whereas a phase transition occurs for $n = 1$, with the Ising-type value for the exponent ν. Approaching two dimensions, the model is shown to tend to the corresponding nonlinear σ-model S_{n-1}. The value of the exponent ν compares well with the first-order $d-2$ expansion. An extension due to Golner, in the scalar case, of the recursion formula and giving an exponent $\eta \neq 0$, is generalised to the vectorial case. It is solved numerically for $2 \leqslant d \leqslant 4$. In three dimensions the value obtained for η is the same within the error bars as the one computed by field theoretic techniques or high-temperature expansions. The value of η in the Ising case, when $d = 2$, compares well with the Onsager value.

1. Introduction

The study of the n-component φ^4 model and the corresponding nonlinear σ-models has played a central role in the modern theory of critical phenomena. They have allowed rather precise determination of critical exponents by means of various perturbative techniques, either in the space dimension d ($4-d$ or $d-2$ expansion) or high-temperature expansion. In the study of more complicated models, like the matrix nonlinear σ-models, such techniques are rarely available or are of unchecked validity. In such a context, it appears useful to develop more global approaches to the critical behaviour. In fact, when he introduced the RG technique to handle the critical problem, Wilson [1], using a phase-space cell analysis, introduced an approximate recursion formula, which he solved numerically for the scalar case in three dimensions, from which he obtained rather good values for the Ising critical exponents, except the exponent η, which was taking the value 0 in his approximation. This last defect was corrected by Golner [2] who introduced an improved recursion formula which produced a good value for the exponent η of the Ising model in three dimensions. To our knowledge at least, since that time there has been no systematic study of the Wilson formula for the vectorial case, and in two dimensions, where the number of components of the vector is expected to play a crucial role. We have undertaken such an analysis and found the critical behaviour for general n in three dimensions. We have also studied these models when the dimension approaches two. We find that there is no phase transition in the non-Abelian case for $n \geqslant 3$ or, in other words, asymptotic

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freedom, in agreement with analytical studies. In the Ising case ($n = 1$), we find a transition, with a value for the exponent ν in reasonable agreement with the Onsager value. Our study also shows in a very neat way how the φ^4 model near the critical point is transformed into the nonlinear σ-model S_{n-1} when the dimension approaches two. It also allows one to compare the results with those of the $d-2$ expansion, which appear to be correct to first order, but of limited validity.

Despite these astonishing successes of the Wilson formula, it suffers from the drawback of giving $\eta = 0$. In order to get a non-zero value for this exponent, we have generalised Golner's approach to the vectorial case and improved its determination of some dimensionality-dependent constants. The recursion formula becomes much more involved in this case. Using rotational invariance, it can be reduced to the study of the transformation of three functions. We have solved these equations numerically in three dimensions and also near two dimensions. In three dimensions, the values obtained for the exponent do not differ too much from those given by the Wilson formula, but the values obtained for η when they are compared with those given by the most sophisticated field theoretic techniques or high-temperature series are in impressive agreement. In two dimensions, for the Ising case, there is good agreement with the Onsager value. On the other hand, when we approach two dimensions we do not get fixed points for systems with a continuous symmetry $n \geqslant 2$.

It is worth pointing out that the Wilson recursion formula is exact when the usual lattice Laplacian is replaced by a hierarchical one. For the Golner recursion formula, there does not seem to exist a hierarchical-type Hamiltonian for which it becomes exact.

In conclusion, it seems that non-perturbative, although approximate, approaches to critical behaviour, like that of Wilson and its extension by Golner are rather powerful tools when applied to vectorial nonlinear σ-models. One can therefore reasonably expect them to prove useful in the analysis of more complicated models, like Grassmannian nonlinear σ-models, which were the original motivation of our study. One should not, however, underestimate the computational difficulties that such extensions would present since, from this point of view, the vectorial models that we have studied are already highly non-trivial.

2. The Wilson formula

2.1. Introduction

The goal of the Wilson formula is to perform a sharp but simple approximation of the renormalisation group (RG). It was derived by Wilson [1] by using a functional integration method and a division of space in cells. Polyakov [3] gave a derivation of this formula by using the Feynman graph expansion, closer in spirit to the RG as applied in field theory. On the other hand, as was first realised by Baker [4], the Wilson formula becomes exact for the hierarchical model introduced previously by Dyson [5, 6]. In such a model the usual discrete Laplacian Δ is replaced by an operator which is such that its inverse displays the same long-distance behaviour as Δ^{-1} and decimates exactly under a RG transformation. Rigorous result about this model have been obtained by Bleher and Sinai [7], Collet and Eckmann [8], and by Gawedzki and Kupiainen [9].

Within this approximation, we consider a family of Hamiltonians

$$H = \int \mathrm{d}^d x \, \tfrac{1}{2}(\nabla \varphi)^2 + Q(\varphi) \tag{1}$$

and the Wilson formula is a recursion equation for the new potential Q' at the new scale $x' = x/L$

$$Q'(\varphi) = -L^d \ln \frac{I(\varphi)}{I(0)}\bigg|_{\varphi \to L^{-\alpha/2}\varphi} \tag{2}$$

with

$$I(\varphi) := \int \mathrm{d}y \exp(-y^2 - Q^+(\varphi, y)) \qquad Q^+(\varphi, y) := \tfrac{1}{2}(Q(\varphi + y) + Q(\varphi - y)) \tag{3}$$

$$\alpha = d - 2 \qquad L = 2. \tag{4}$$

To compare with the usual $\lambda\varphi^4$ RG, which is done perturbatively in λ and in $\varepsilon = 4 - d$ and where we calculate the lowest-order graphs exactly, the diagrammatic derivation of Polyakov shows that the new potential already includes an infinite number of graphs, but not all the possible graphs, and each included graph is approximated. Thus, we have another point of view on the RG. Moreover, in (2) the spatial dimension d appears as a parameter, giving us the occasion to have solutions non-perturbatively with the dimension. In deriving (2) we neglect all dependence on $\nabla\varphi$, which makes the formulae simple, but we always have $\alpha = d - 2$ or

$$\eta \equiv 0 \tag{5}$$

which means that the anomalous dimension of the field is identically zero.

This formula is not 'unique', in the sense that we could take other forms for Q^+, for example the non-symmetrised form $Q^+ = Q(\varphi + y)$ used by Gallavotti. Nevertheless, this expression for Q^+ is natural for a RG realised as a modification of the cutoff in k space. There is still the parameter L in the formulae, which could *a priori* be chosen in $]1, \infty[$, but the two terms on the right in the definition of Q^+ are linked to the choice $L = 2$. As already discussed by Wilson in [1], this value is a good compromise between 'no coarse graining' ($L = 1$) and 'too much coarse graining' ($L \to \infty$). In this work, we will always take $L = 2$, but retain the L dependences up to the final formulae.

Now, we are interested in second-order phase transitions, i.e. the fixed points of the Wilson formula. We can quickly check that $Q = 0$ constitutes a trivial fixed point of (2). In searching for a non-trivial one, we can follow the text book calculation for the potential $Q(\varphi) = m\varphi^2 + \lambda\varphi^4$ and calculate perturbatively in $\varepsilon = 4 - d$. At the lowest order, we find the usual critical exponent for the mass

$$\nu = \frac{1}{2} + \frac{\varepsilon}{4}\frac{n+2}{n+8}. \tag{6}$$

In the next two subsections we will explore other ways of finding non-trivial fixed points of (2), namely a $2 + \varepsilon$ expansion and a systematic numerical study.

2.2 Expansion in $d = 2 + \varepsilon$

The physical idea which is behind the $d = 2 + \varepsilon$ expansion is that a system with a continuous symmetry undergoes a phase transition for a temperature $T = \mathrm{O}(\varepsilon)$, and that the effective potential concentrates the vector φ on a sphere of radius 1. In other words, close to two dimensions, $\beta \to \infty$ and the vectorial model $\mathbb{R}^n$ tends to a nonlinear σ-model $S_{n-1} = \mathrm{O}(n)/\mathrm{O}(n-1)$. The difficulty of this calculation is that, for $d \to 2$, the weight $\exp(-Q(\varphi))$ tends to $\delta(\varphi^2 - 1)$, which is singular. Hence, by contrast with the $4 - \varepsilon$ expansion, this calculation is not a perturbation of a trivial fixed point. This idea

has already been used by Gawedzki and Kupiainen [9] to prove the asymptotic freedom of the S_n model in dimension 2, but they used the non-symmetrised form for Q^+. To compare with the method used in field theories for the nonlinear σ-model, the present method could be called 'extrinsic' because the sphere S_{n-1} is simulated by a potential in the bigger space $\mathbb{R}^n$. The usual 'intrinsic' method is to eliminate the constraint induced by the manifold, i.e. to introduce a coordinate system on the sphere. By doing that, we lose the thickness of the manifold. We will come back to this method when discussing the Golner equations.

As temperature plays an important role, let start with the Hamiltonian

$$\beta H(\phi) = \int \mathrm{d}^d x[\beta \tfrac{1}{2}(\nabla\phi)^2 + \beta v(\phi^2 - 1)] \tag{7}$$

and the potential $v(x)$ has a very strong minimum when the argument is zero, i.e. for $\phi^2 = 1$. We choose the normalisation constant of the potential such that $v(0) = 0$. With the rescaling $\sqrt{\beta}\phi \to \varphi$ and the definition $Q(x) := \beta v(x/\beta)$, we are left with the standard Hamiltonian

$$H = \int \mathrm{d}^d x[\tfrac{1}{2}(\nabla\varphi)^2 + Q(\varphi^2 - \beta)] \tag{8}$$

and now the potential shows a strong minimum around $\varphi^2 = \beta$. We define

$$f(x) := \exp(-Q(x)/2) \tag{9}$$

and the condition for the minimum is

$$\partial f(0) = \left.\frac{\partial f}{\partial \varphi^2}\right|_{\varphi^2=\beta} = 0. \tag{10}$$

By rotational invariance of the Hamiltonian, we can take φ in the first direction. We denote the vector y by $y = (y_\parallel, y_\perp \cdot \boldsymbol{e}_\perp)$ and $\boldsymbol{e}_\perp$ is a unit vector perpendicular to φ. With this notation, the Wilson formula becomes

$$\begin{aligned}(f'(\varphi^2 - \beta'))^{2L^{-d}} &= C \int_{-\infty}^{\infty} \mathrm{d}y_\parallel \int_0^{\infty} \mathrm{d}y_\perp \, y_\perp^{n-2} \exp(-y_\perp^2 - y_\parallel^2) f(\varphi^2 + y_\perp^2 + y_\parallel^2 + 2\varphi y_\parallel - \beta) \\ &\quad \times f(\varphi^2 + y_\perp^2 + y_\parallel^2 - 2\varphi y_\parallel - \beta)|_{\varphi \to L^{-\alpha/2}\varphi}.\end{aligned} \tag{11}$$

The new variable β' is *defined* by the condition

$$\partial f'(0) = \left.\frac{\partial f'(\varphi^2 - \beta')}{\partial \varphi^2}\right|_{\varphi^2=\beta'} = 0. \tag{12}$$

This condition means simply that the minimum of f' is at zero, in other words we stay in the set of the considered functions. We define the new variable

$$x := \varphi^2 - L^\alpha \beta \tag{13}$$

and we suppose that x is $O(\beta^0)$

$$\varphi = \sqrt{L^\alpha \beta + x} \simeq L^{\alpha/2}\sqrt{\beta}. \tag{14}$$

It is more convenient to work with the shift of the minimum $a = O(\beta^0)$ defined by

$$x + L^\alpha a := \varphi^2 - \beta' \tag{15}$$

and we have the relation between β, β' and a

$$\beta' = L^{d-2}\beta\left(1-\frac{a}{\beta}\right) \tag{16}$$

where we used $\alpha = d-2$ (and $\eta = 0$). Introducing this definition and approximation in (11), making the change of variable $2L^{\alpha/2}\sqrt{\beta}\,y_\parallel \to y_\parallel$, and supposing that $f(y_\parallel)$ decays faster than $\exp(-y_\parallel^2/\beta)$ for $\beta \to \infty$, the recursion equation for f becomes

$$(f'(x+a))^{2L^{-d}} = C\int_{-\infty}^{\infty} dy_\parallel \int_0^{\infty} dy_\perp\, y_\perp^{n-2} \exp(-y_\perp^2) f(L^{-\alpha}x + y_\perp^2 + y_\parallel)$$
$$\times f(L^{-\alpha}x + y_\perp^2 - y_\parallel) \tag{17}$$

with the conditions $\partial f(0) = 0$ and $\partial f'(0) = 0$, the second one fixing a, together with the equation (16) for β' and the constant being fixed by the constraint $f'(0) = 1$. It is not at all obvious that a fixed point of this equation exists.

When a fixed point of (17) exists, equation (16) is of the same form as that given, to lowest order in $d-2$, by field theoretic techniques. It has a fixed point $\beta^* = a/(d-2)\ln 2$ if $a > 0$ with the corresponding exponent $\nu = 1/(d-2)$, which is the standard value, to lowest order in ε. It remains to compute a. We do this approximately. Thus, we take the simplest function Q with the right properties

$$Q(x) = \lambda x^2. \tag{18}$$

To discuss this fixed point, it will be convenient to introduce the auxiliary function J_n

$$J_n(x) := \int_0^{\infty} dy\, y^n \exp(-y^4 - xy^2). \tag{19}$$

With an integration by parts, we obtain

$$J_{n+4} = \frac{n+1}{4} J_n - \frac{x}{2} J_{n+2} \tag{20}$$

and

$$\frac{\partial J_n(x)}{\partial x} = -J_{n+2}. \tag{21}$$

With this simple form for Q, we can perform the integration on $y_\parallel$ in equation (17), and obtain the new potential

$$L^{-d}Q'(x + L^\alpha a) = \lambda x^2 - \ln J_{n-2}\left(\frac{2\lambda x + 1}{\sqrt{\lambda}}\right)\Bigg|_{x \to L^{-\alpha}x}. \tag{22}$$

The value of a is fixed by the condition $\partial Q'(0) = 0$ at $x = -L^\alpha a$

$$a = \frac{1}{\sqrt{\lambda}} \frac{J_n}{J_{n-2}}\left(\frac{1-2\lambda a}{\sqrt{\lambda}}\right). \tag{23}$$

The next term of the Taylor expansion of Q' around $-L^\alpha a$ gives λ'

$$L^{-d}\frac{\partial^2 Q'}{\partial x^2} = 2\lambda - 4\lambda\left(\frac{J_{n+2}}{J_{n-2}} - \left(\frac{J_n}{J_{n-2}}\right)^2\right). \tag{24}$$

By using the recursion equation for J_{n+2} and the equation for a, we find

$$\lambda' = L^d \lambda \left(a - \frac{n-3}{2} \right) \tag{25}$$

and the new potential at second order is $Q'(x - L^a a) = L^d \lambda (a - \frac{1}{2}(n-3))(x - L^a a)^2$. At the fixed point, the condition $\lambda' = \lambda$ gives an equation for a that could be solved

$$a = \tfrac{1}{2}(n - 3 + 2L^{-d}) = \tfrac{1}{2}(n - \tfrac{5}{2}). \tag{26}$$

This result can be compared with field theory (see for example the book by Amit [10]) $a = n - 2$. The fact that the coupling constant $t^* = 1/\beta^* \to 0$ when $d \to 2$ is interpreted in field theory as asymptotic freedom in ultraviolet at dimension 2. In statistical mechanics we say that the phase transition temperature is going to zero.

In this approximation we see that Wilson formula predicts no phase transition in $d = 2$, when $n \geqslant 3$, in agreement with the exact result, obtained perturbatively in $d - 2$. The numerical results presented below will confirm this prediction. The case $n = 2$, where the topological phase transition of Kosterlitz and Thouless should occur, seems to be out of reach of the Wilson formula. On the other hand, when $n = 1$ this formula reproduces the results of the Ising model.

This calculation could also be done with the variable ϕ and the Hamiltonian (7), but with some more care because

$$\beta \exp[-\beta v(\phi^2 - 1)/2] = \beta f(\beta(\phi^2 - 1)) \xrightarrow{\beta \to \infty} C\, \delta(\phi^2 - 1). \tag{27}$$

This shows that f is a function that could be regular, and a singular distribution $\delta(\phi^2 - 1)$ for ϕ is obtained in the above limit.

This calculation shows that the Wilson formula has kept the essential part of the RG, even in such a difficult case as the $2 + \varepsilon$ expansion. This, and the usual $4 - \varepsilon$ expansion, will be well illustrated by numerical solutions presented in the next subsection. In particular, this shows how the system, at the fixed point, goes from a vector-like model close to four dimensions to a nonlinear σ-model close to two dimensions.

2.3. Numerical resolution of the Wilson formula

2.3.1. Numerical problems. The RG transforms a problem of phase transition into the study of a flow of Hamiltonians, i.e. it transforms a problem where the correlation length diverges in another one where the computational time diverges. Thus, it is essential to find an efficient algorithm to keep the computational time in reasonable bounds. Moreover, we want to find fixed points for different dimensions and numbers of components with a more efficient method than the original one used by Wilson. For the discussion, we will keep in mind the picture presented in figure 1.

One of the difficulties is that the fixed point is of codimension one, but we will use this fact to construct an efficient algorithm. The principle of the method is to take a starting potential Q_0 and iterate the Wilson transformation. First, we will go in the direction of the searched-for fixed point, and then along the unstable manifold to the low- or high-temperature fixed point. Because the iteration is going exponentially fast, after a sufficient number of iterations we can decide on which direction we are driven by the flow of the RG, and choose a new starting potential closer to the stable manifold.

One of the obstacles to turning this idea into an algorithm is that there is no natural metric on the space of Hamiltonians. Several 'norms' have been tried (strictly they

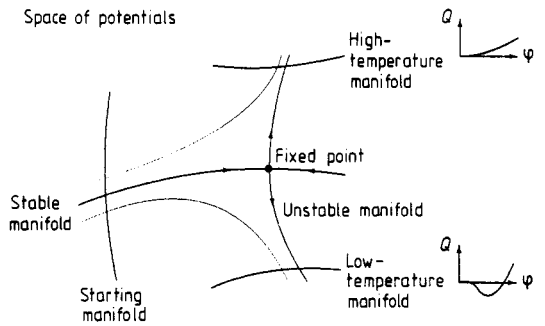


Figure 1. Spatial relationships between the different manifolds and the corresponding renormalisation group trajectories to the fixed point.

are not norms because they do not satisfy the triangular inequality). Close to the fixed point, they give equivalent results, but far from it we have problems. After many attempts, a good choice is

$$\|x(\varphi)\|^2 = \int \mathrm{d}\varphi \exp(-\varphi^2 - Q(\varphi))x^2(\varphi)\left(\int \mathrm{d}\varphi \exp(-\varphi^2 - Q(\varphi))\right)^{-1} \tag{28}$$

and the distance between Q and its iterate Q' is given by $\|Q - Q'\|$. Now with this distance, we can measure how close we are from the fixed point.

For the starting manifold, we take two potentials Q_0 and Q_1, such that the iterations of Q_0 cross the high-temperature manifold and those of Q_1 cross the low-temperature manifold. The linear combination $Q_\alpha = \alpha Q_1 + (1-\alpha)Q_0$ needs to cross the stable manifold. Thus, we can surround the intersection of the two manifolds by Q_{α_0} and Q_{α_1}. Moreover, if the distance between the jth and the $(j+1)$th iterations of Q_α is decreasing for j between 1 and a j_{fp}, that means that we are approaching the fixed point and we can take as the new functions Q_0 or Q_1, the j_{fp} iteration of Q_α. This process is illustrated in figure 2.

In that way, the starting manifold approaches the fixed point by following the flow of the RG and uses to its advantage the dilating direction of the flow. At the initialisation, we take Q of the form $Q_0 = m_0\varphi^2 + \lambda\varphi^4$ and $Q_1 = m_1\varphi^2 + \lambda\varphi^4$. This way of working is particularly suitable when the manifold $m\varphi^2 + \lambda\varphi^4$ is far from the fixed point, for example when the dimension is close to 2.

Another difficult problem is the choice of the stopping manifold, i.e. giving a criterion to decide that the flow is driving us to the low- or high-temperature region.

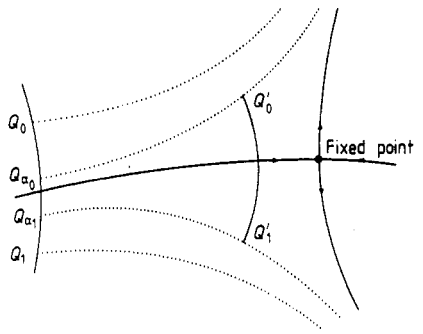


Figure 2. Choosing the new functions Q_0' and Q_1' as the iterations approach the fixed point.

In fact, it is difficult to give in a simple way a manifold which is only on one side of the stable manifold, or at least on a big enough part of the space of potential close to the fixed point. After several attempts, we took the manifolds

$$I = \ln\left(\frac{\int \mathrm{d}\varphi \, \exp(-\varphi^2 - Q(\varphi))}{\int \mathrm{d}\varphi \, e^{-\varphi^2}}\right) \tag{29}$$

low-temperature manifold: $I = I_{\min}$ high-temperature manifold: $I = I_{\max}$.

At the beginning, we place these manifolds far from the estimated fixed point. If, after having passed close to the fixed point, the number of iterations up to the crossing of the stopping manifold is big enough, we can safely move them towards the fixed point.

Thus, when it approaches the fixed point, the program will move the starting and stopping manifolds towards it, and so focus in the space of potentials around the fixed point, exactly in the same fashion as the RG does. With this algorithm, the number of iterations needed to find the fixed point at a given precision is kept small.

However, when we go close to two dimension, the potential has a strong minimum and $\exp(-Q(\varphi))$ becomes big enough to create overflows in the computation of the integrals. This is an important problem on a VAX, where the exponents are limited to 37. To solve this, we subtract the minimum of $\varphi^2 + Q(\varphi)$ to the potential in the computation of the integrals. This is of no importance for the new potential, and we need to add $\min(\varphi^2 + Q(\varphi))$ to I or $\ln\|Q_j - Q_{j+1}\|$ (this explains why we take logarithms in this function).

The exponent ν is estimated by plotting $\ln\|Q_j - Q_{j+1}\|/\ln L$ as a function of j. After the minimum of the curve, we fit a straight line by a least-squares method and the slope gives us $1/\nu$.

Numerically, the potential $Q(\varphi)$ is given on a regular mesh up to a value $\varphi_{\max}$. Between the points of the mesh, Q is interpolated by $a + b\varphi + c\varphi^2$. To avoid numerical short-distance noise on Q, the interpolating function is fitted by a least-squares method on a fixed length (smoothing length). Beyond $\varphi_{\max}$, Q is extrapolated by $a + c\varphi^{\alpha}$ and the parameters a, c and α are fitted by a least-squares method on the smoothing length. This is a non-trivial fit because α appears nonlinearly in the extrapolating function.

By using the rotational invariance of the Hamiltonian, we can reduce the n-dimensional integral on y to a two-dimensional integral where n appears as a parameter. The two-dimensional integrals are computed by the standard Simpson algorithm and a part of the program needs to be adapted to the special case $n = 1$.

2.3.2. Numerical results. We are left with two parameters in the problem, namely the number of components of the model n, and the dimension of space d. At the fixed point, it is interesting to know the potential $Q(\varphi)$, the critical exponent $1/\nu$ and the probability of φ, i.e.

$$P(\varphi) = \mathrm{C}\, \varphi^{n-1} \exp(-\varphi^2 - Q(\varphi)) \tag{30}$$

and the constant is fixed by $\int P(\varphi)\, \mathrm{d}\varphi = 1$. We choose the following measurements:

for $n = 5$	$2 \leqslant d \leqslant 4$	$Q, 1/\nu$
for $n = 1$	$2 \leqslant d \leqslant 4$	$Q, 1/\nu$
for $d = 3$	$1 \leqslant n \leqslant 20$	$P, 1/\nu$
for $d = 2.1$	$1 \leqslant n \leqslant 10$	$Q, 1/\nu$.

The graphs are given in figures 3-11, and we will comment on the results here.

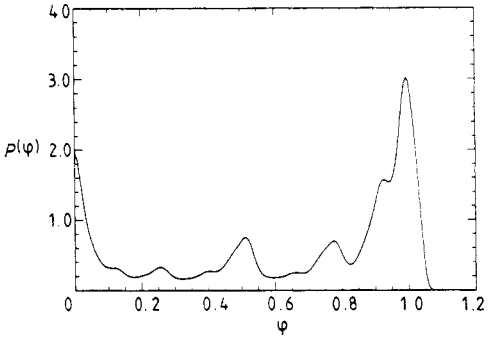


Figure 3. Probability P of φ for the one-component model in dimension $d = 2.1$.

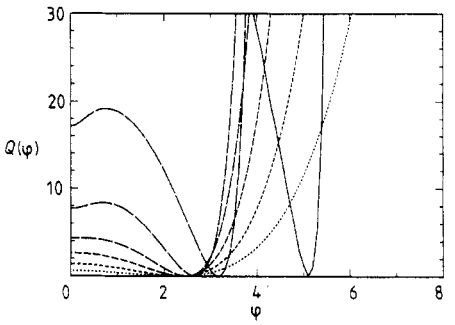


Figure 4. Potential $Q(\varphi)$ at the fixed point for the five-component model, with the convention that the potential is zero at the minimum. In order of increasing dash size, we have the dimensions $d = 3.6$, 3.3, 3.0, 2.75, 2.3, 2.1 (full curve).

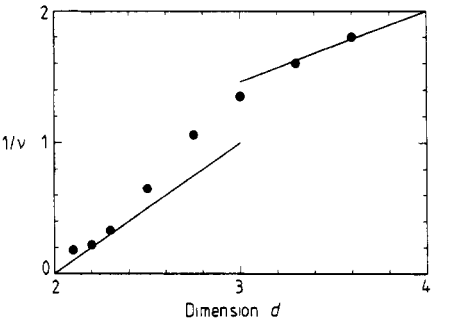


Figure 5. Critical exponent $1/\nu$ as a function of spatial dimension for the five-component model. The full circles display the numerical results, the lower-left straight line is the $2+\varepsilon$ expansion, the upper-right line is the $4-\varepsilon$ expansion, both to first order in ε.

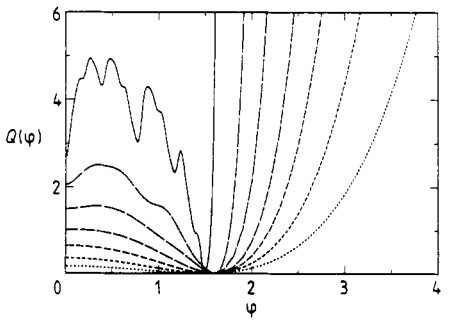


Figure 6. Potential $Q(\varphi)$ at the fixed point for the one-component model, with the convention that the potential is zero at the minimum. In order of increasing dash size, we have the dimensions $d = 3.6$, 3.3, 3.0, 2.75, 2.5, 2.3, 2.1 (full curve).

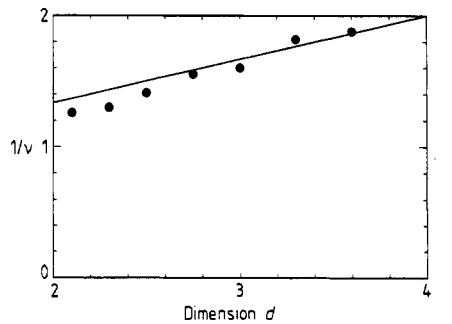


Figure 7. Critical exponent $1/\nu$ as a function of spatial dimension for the one-component model. The full circles represent the numerical results, the straight line represents the $4-\varepsilon$ expansion to first order. The Onsager solution for the Ising model at two dimensions gives $1/\nu = 1$.

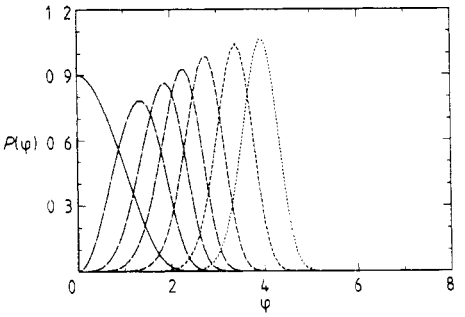


Figure 8. Probability P of φ at the fixed point in three dimensions. In order of decreasing dash size, we have the models $n = 1$ (full curve), 3, 5, 7, 10, 15, 20.

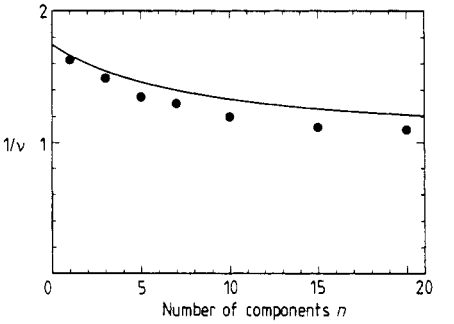


Figure 9. Critical exponent $1/\nu$ at dimension 3 plotted against the number of components of the model. The full circles represent the numerical results. The full curve denotes the $4-\varepsilon$ expansion to first order.

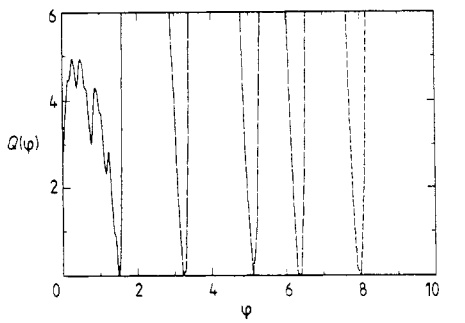


Figure 10. Potential $Q(\varphi)$ at the fixed point in 2.1 dimensions. In order of decreasing dash size, we have the models $n = 1$ (full curve), 3, 5, 7, 10.

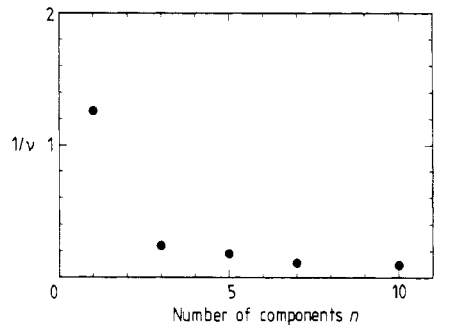


Figure 11. Critical exponent $1/\nu$ at dimension 2.1 plotted against the number of components of the model.

The curves for the one- and five-component models show well how we go *continuously* with the dimension from a nonlinear σ-model $O(n)/O(n-1) = S_{n-1}$ for $d \to 2$ to a φ^4-like model for $d \to 4$. This shows why these models should be in the same universality class. Moreover, this illustrates the common lore that close to the lower critical dimension (the dimension below which there is no ordered phase whatever the temperature is), the phase transition $O(n) \to O(n-1)$ is dominated by angular fluctuations (the direction of $\boldsymbol{\varphi}$) and the 'longitudinal fluctuations' (the norm of φ) is frozen, i.e. the phase transition is described by the nonlinear σ-model S_{n-1}.

The usual field theory calculation keeps only the *a priori* relevant terms, and for the calculation close to two dimensions it goes directly to the S_{n-1} model, i.e. it takes a distribution like $\delta(\varphi^2 - 1)$. On the other hand, if we think of the RG in statistical mechanics as decimation, i.e. as averaging the field over some distance, it appears natural that by averaging vectors of length 1 we get a continuous distribution for the average vector (by the way, in order to perform an average we need a least a vectorial space). This fact is well illustrated by $P(\varphi)$ for $n = 1$ in dimension 2.1 given below, where we rescale φ so that the last maximum of $P(\varphi)$ is at $\varphi = 1$.

Close to two dimensions, the one-component model tends to the S_0 model, i.e. to a model with the two values ± 1, or the well known Ising model. By performing a block spin average 'à la Kadanoff' with two Ising spins, we obtain successively the set

of values ± 1, $\pm 1\ 0$, $\pm 1\ \pm\frac{1}{2}\ 0$, $\pm 1\ \pm\frac{3}{4}\ \pm\frac{1}{2}\ \pm\frac{1}{4}\ 0$, etc ... and we see that the maxima of $P(\varphi)$ correspond to those values. But qualitatively it is what the Wilson formula does, for $L=2$. We expect that at two dimensions, the fixed point is described by a singular distribution.

Again close to two dimensions, but for the five-component model, we see that the minimum of the potential is going in the direction of large φ, which is equivalent to $\beta \to \infty$ if we make the choice of normalisation of φ such that the potential is minimum at $\varphi = 1$.

The critical exponent $1/\nu$ behaves as expected, i.e.

(i) the $4-\varepsilon$ expansion is valid;

(ii) for the five-component model, the $2+\varepsilon$ expansion is valid and there is no phase transition at $d=2$

(iii) for the one-component model, there is a phase transition at $d=2$.

We can also note that the $4-\varepsilon$ expansion at first order is valid up to $\varepsilon = 1$, and even for the one-component model up to $\varepsilon = 2$. The latter is surprising because the potential is no longer φ^4-like. This contrasts with the $2+\varepsilon$ expansion at first order, which is not valid so far.

The graphs for three dimensions show that

(i) the exponent $1/\nu$ has the right dependence on n;

(ii) for large n, $P(\varphi)$ goes to a steep curve, but slowly. Thus, the $1/n$ expansion from $n=\infty$ will have bad convergence properties up to $n = O(1)$.

Finally, for dimension 2.1, the critical exponent $1/\nu$ shows that the one-component model has a phase transition at $d=2$, but there is no phase transition for $n \geq 3$. This compares well with the $d=2+\varepsilon$ expansion above, which puts the change of behaviour at $n=2.5$. This is equivalent to the statement that at two dimensions the models with $n \geq 3$ are asymptotically free.

3. The Golner recursion formula

One problem with the Wilson formula is that it gives $\eta \equiv 0$. Golner proposed an approximation in the same spirit of that of Wilson, which gives an exponent $\eta \neq 0$. The problem is to include lower-order corrections to the terms in $\nabla\varphi$ in the Hamiltonian, which are responsible for the small value of η. Golner [2] derives this formula for the scalar case.

3.1. Derivation of the formula by real space approximations

The derivation presented here for the vectorial case mainly follows the method of Golner, which is an extension of the original method due to Wilson, in which the real space is divided into cells (weakly coupled). However, this extension for the vectorial model is not straightforward; therefore we will give the main idea below. We start with the Hamiltonian

$$H(\varphi) = \int \mathrm{d}^d x \frac{1}{2\gamma} \nabla\varphi W(\varphi) \nabla\varphi + \omega Q(\varphi) \tag{31}$$

with $\varphi \in \mathbb{R}^n$, $W(\varphi^2)$ is a matrix of dimension $n \times n$, typically of the form $\mathbb{1} + O(\varphi^2)$ where the term $O(\varphi^2)$ small, γ, ω are two constants; ω will be fixed by the inverse of the volume of a cell.

We denote by $i, j, k \in (1, \ldots, d)$ the space indices and by $\beta, \gamma, \mu, \nu \in (1, \ldots, n)$ the indices of φ.

Roughly, the idea of the RG is to derive a new Hamiltonian at scale x/L by

$$e^{-H'(\varphi')} = \int dy \exp(-H(\varphi' + y)) \tag{32}$$

where $y(x)$ represents the fluctuations at small length scale (or large k) and $\varphi'(x/L)$ represents the new field, with fluctuations at the new scale x/L (for the details see the review of Wilson and Kogut [7]). In order to derive an approximate RG, we divide the space in cells and decompose $\varphi(x)$ on them

$$\varphi(x) = \sum_{\text{cell}} \psi_m(x) y_m + L^{-\alpha/2} \varphi'(x/L). \tag{33}$$

The function $\psi_m(x)$ gives the fluctuations at the scale of one cell, as shown in figure 12. The volume of the cell (block) is fixed by $\int_{x \in B} d^d x = 1/\omega$, and $\psi(x)$ is normalised by $\int_{x \in B} |\psi(x)|^2 = 1$ and by $\int_{x \in B} \psi(x) = 0$. Now comes the extension with respect to the Wilson derivation of an approximate RG. We take φ' 'almost' constant at the cell scale

$$\varphi'(x/L) = \varphi'(x_0/L) + (x - x_0)_i \partial_i \varphi'(x_0/L) + \tfrac{1}{2}(x - x_0)_i (x - x_0)_j \partial_i \partial_j \varphi'(x_0/L) \tag{34}$$

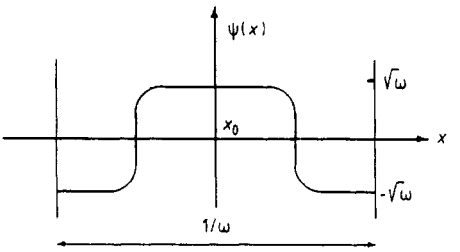


Figure 12. The function $\psi(x)$.

where x_0 is the centre of the cell. The new terms $\nabla\varphi'(x/L)$ and $\nabla^2\varphi'(x/L)$ are small compared to $\varphi'(x/L)$ and give a coupling between different cells at the new scale x/L, whereas we neglect all couplings between different cells at scale x

$$\int d^d x H[\varphi(x)] = \sum_m \int_{x \in B_m} d^d x H(\psi_m(x) y_m + L^{-\alpha/2} \varphi'(x/L)) \tag{35}$$

with B_m the cell (block) number m and $x/L \in B_m$. We obtain for the new Hamiltonian

$$\exp\left(-\int d^d x H'[\varphi]\right) = \prod_m \int dy \exp\left(-\int_{x \in B_m} d^d x\, H(\psi_m(x) y + \varphi'(x/L))\right)\Big|_{\varphi'(x/L) \to L^{-\alpha/2}\varphi(x);\, x/L \to x} \tag{36}$$

and we can limit ourselves to calculating integrals in one cell. In order to be consistent with the approximations made on φ', we want to work at order ∇^2. Now a long calculation takes place, and we give here directly the recursion relation for the rescaled

functions W and Q

$$W'_{\beta\gamma}(\varphi) = L^{d-2-\alpha}\left(\langle W^+_{\beta\gamma}(\varphi, y)\rangle + \frac{1}{2}\left\langle \frac{\partial}{\partial\varphi_\beta}(\rho_2 y W(\varphi) y + CQ^+) \cdot \frac{\partial}{\partial\varphi_\gamma}(yW(\varphi)y + Q^+)\right\rangle_c\right) \tag{37}$$

$$Q'(\varphi) = -L^d \ln \frac{I(\varphi)}{I(0)} \qquad \text{at } \varphi \to L^{-\alpha/2}\varphi.$$

α is fixed by

$$W'(0) = W(0). \tag{38}$$

We use the following definition:

$$\begin{aligned} Q^+(\varphi, y) &= \tfrac{1}{2}(Q(\varphi + y) + Q(\varphi - y)) \\ I(\varphi) &= \int \mathrm{d}^n y \exp(-yW(\varphi)y - Q^+(\varphi, y)) \\ \langle A(\varphi, y)\rangle &= \frac{1}{I(\varphi)} \int \mathrm{d}^n y \exp(-yW(\varphi)y - Q^+(\varphi, y)) A(\varphi, y). \end{aligned} \tag{39}$$

We note by $\langle\,\rangle_c$ the cumulant expansion and we separate the variables by a dot $\cdot$ if confusion is likely

$$\begin{aligned} \langle A\rangle_c &= \langle A\rangle \\ \langle A \cdot B\rangle_c &= \langle AB\rangle - \langle A\rangle\langle B\rangle \\ &\vdots \end{aligned}$$

The arbitrary constant added to H' has been fixed such that $Q(0) = 0$, but any other choice is possible. The condition $W'(0) = W(0)$ fixes the value of α, and we write as usual $\alpha = d - 2 + \eta$, in which η is the anomalous dimension of the field. In the present work, if not otherwise stated, we will take $W(0) = \mathbb{1}$.

We recover the Wilson formula if we take $W = \mathbb{1}$ and if we neglect the term $\langle \partial Q^+/\partial\varphi \cdot \partial Q^+/\partial\varphi\rangle_c$. For W' we obtain $W' = L^{d-2-\alpha}\,\mathbb{1}$ with the equation for α: $W'(0) = \mathbb{1}$. They are satisfied for $\alpha = d - 2$ and so for the Wilson formula we always have $\eta \equiv 0$. This shows that in order to obtain $\eta \neq 0$, we need to take into account terms with the fluctuations of the potential.

A trivial fixed point of the recursion equation is

$$W = \mathbb{1} \qquad Q = 0 \qquad \alpha = d - 2. \tag{40}$$

We recover here the Gaussian fixed point.

A last remark: we notice that during the derivation of the recursion equations, we never used the particular structure of the field φ nor the symmetry of the Hamiltonian, except the discrete symmetry $H(-\varphi) = H(\varphi)$. Thus, the recursion equations are valid for vectorial models as well as for matrix or other models.

3.2. Evaluation of the constants

We would like to evaluate the various constants ω^{-1}, V, ρ_0, ρ_2 and C which appear during the derivation of the recursion equations. For this, we will make 'physical' assumptions about the elementary cell and about the fluctuation function $\psi(x)$. Here

we proceed differently to Golner [2] who makes other assumptions in reciprocal space. We suppose that the elementary cell is spherical with a radius R

$$\omega^{-1} = \int_{\text{cell}} \mathrm{d}^d x = \Omega_{d-1} \frac{R^d}{d} \tag{41}$$

and Ω_d denote the surface of the sphere S_d at d dimensions (the general formula for $d \in \mathbb{R}$ is $\Omega_d = 2\pi^{(d+1)/2}/\Gamma[(d+1)/2]$). In the same fashion

$$V = \int_{\text{cell}} \mathrm{d}^d x\, x_i^2 = \frac{\Omega_{d-1}}{d} \int_0^R \mathrm{d}r\, r^{d+1} = \frac{\Omega_{d-1}}{d} \frac{R^{d+2}}{d+2}. \tag{42}$$

To evaluate ρ_0 and ρ_2, we suppose $\psi(x)$ has spherical symmetry, as shown in figure 13, and $|\nabla\psi|^2 \simeq \omega\delta(r-\tilde{R})$. The value of $\tilde{R}$ is given by the condition $\int_{\text{cell}} \psi(x) = 0$

$$\tilde{R} = \frac{R}{\sqrt[1/d]{}}. \tag{43}$$

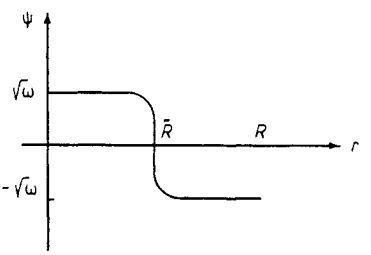


Figure 13. A spherically symmetric form of $\psi(r)$.

With the approximations

$$\rho_0 = \int \mathrm{d}^d x |\nabla\psi|^2 = \Omega_{d-1}\omega \int \mathrm{d}r\, r^{d-1}\delta(r-\tilde{R}) = \omega\Omega_{d-1}\tilde{R}^{d-1} \tag{44}$$

$$\rho_2 = \int \mathrm{d}^d x\, x_i^2 |\nabla\psi|^2 = \frac{\Omega_{d-1}}{d} \int \mathrm{d}r\, r^{d+1}\delta(r-\tilde{R}) = \omega\Omega_{d-1}\frac{\tilde{R}^{d+1}}{d} \tag{45}$$

we can calculate C and ρ_2

$$C = V\omega\rho_0 = \frac{d}{d+2}\frac{R}{2^{d-1/d}} \tag{46}$$

$$\rho_2 = \frac{R}{2^{d+1/d}} = \frac{d+2}{2^{2/d}d} C. \tag{47}$$

For dimensions 2, 3 and 4, the ratio ρ_2/C is respectively 1, 1.05 and 1.06, therefore we can take $\rho_2 = C$. Finally, there is still a constant to fix arbitrarily, and a natural choice is to take $\omega = 1$:

$$C = \frac{d}{d+2}\left(\frac{d}{\omega\Omega_{d-1}2^{d-1}}\right)^{1/d}. \tag{48}$$

For dimensions 2, 3 and 4 we obtain respectively the values $C(2) = 0.20$, $C(3) = 0.23$, $C(4) = 0.27$. We could also make other choices for the arbitrary constant, for example $R = 1$.

3.3. The vectorial model

3.3.1. General form of the Hamiltonian. Now we would like to find the precise structure of the Hamiltonian for the vectorial model. We take $\varphi \in \mathbb{R}^n$ and impose the global invariance under rotation

$$H(R\varphi) = H(\varphi) \qquad \forall R \in O(n). \tag{49}$$

The scalar quantities that could be constructed from φ and $\nabla\varphi$ are φ^2, $(\nabla\varphi)^2$, $(\varphi\nabla\varphi)^2$ and all their polynomials. If we restrict ourselves to the order $(\nabla\varphi)^2$, the Hamiltonian takes the form (31) and the invariance condition becomes

$$Q(R\varphi) = Q(\varphi) \qquad W(R\varphi) = RW(\varphi)R^+. \tag{50}$$

We can easily check that the approximate RG given by (37) preserves the rotational invariance under $O(n)$. The general form for Q and W is

$$Q = Q(\varphi^2) \qquad W = f(\varphi^2)\cdot\mathbb{1} + g(\varphi^2)\cdot\varphi\times\varphi \tag{51}$$

with f, g, Q scalar functions of φ^2 and $f(0) = 1$. These three functions will be evaluated numerically at the fixed point for different values of n and for different dimensions d. For the expansion around the Gaussian fixed point ($4-\varepsilon$ expansion), we approximate these functions by

$$Q = m\varphi^2 + \lambda\varphi^4 + \sum_{j=3}^{\infty} r_j\varphi^{2j} \qquad f = 1 + s\varphi^2 \qquad g = q \tag{52}$$

and we use the shorthand notation $\varphi^{2j} = (\varphi^2)^j$.

On the other hand, the above form for the Hamiltonian (31) and (51) is quite interesting in view of a $d = 2+\varepsilon$ expansion. In fact, when we study the S_n nonlinear σ-model viewed as a vector of length 1 in $\mathbb{R}^{n+1}$, and we eliminate the constraint [11, 12], i.e. we introduce a coordinate system around a pole [13], then we obtain a Hamiltonian of this form. Then after having introduced a magnetic field and using the Golner formula as an approximate RG, we recover the results of Brézin *et al* at lowest order in ε.

3.4. $d = 4-\varepsilon$ expansion

We aim at finding for the vectorial model a non-trivial fixed point for the recursion equations close to the Gaussian fixed point. The standard $d = 4-\varepsilon$ perturbative expansion is used, and in order to simplify the calculations, we introduce at the very beginning the right order in ε for the coefficients of Q and W in (52)

$$m, \lambda \sim O(\varepsilon) \qquad r_j \sim O(\varepsilon^j) \qquad s, q \sim O(\varepsilon^2).$$

For convenience, we gather here all the recursion equations in which we have used

$\alpha = d-2+\eta$ and $\eta = O(\varepsilon^2)$:

$$\begin{aligned} m' &= L^2[m+\lambda(n+2)+(q+ns)/2-m\lambda(n+2)-\lambda^2(n+2)^2] \\ \lambda' &= L^{4-d}[\lambda-\lambda^2(n+8)] \\ r_j' &= L^{d-j\alpha}\left(r_j-\frac{(-2\lambda)^j}{2j}(n-1+3^j)\right) \\ 1 &= L^{-\nu}\left(1+\frac{q+ns}{2}\right) \\ s' &= L^{2-d}(s+8C\lambda^2) \\ q' &= L^{2-d}\left(q+8C\lambda^2\frac{n+6}{2}\right). \end{aligned} \tag{53}$$

The relation $1 = \ldots$ comes from the condition $f(0)=1$ and fixes η. As a particular solution, we recover the Gaussian fixed point $m=\lambda=r_j=s=q=0$ with $\alpha=d-2$, $\eta=0$. We get for the two critical exponents η and ν

$$\nu = \frac{1}{2}+\frac{\varepsilon}{4}\frac{n+2}{n+8} \tag{54}$$

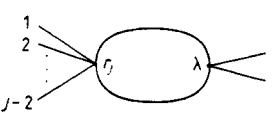
$$\eta = \varepsilon^2\frac{6C\ln L}{L^2-1}\frac{n+2}{(n+8)^2}. \tag{55}$$

These exponents can be compared with the standard field theory result as obtained by Brezin *et al* [14, p 206], with the method of dimensional regularisation. It is quite remarkable that such a simple formula as the Golner one gives, at lowest order, the right dependence in ε, the right dependance in n, an exponent ν exact (as is already the case with the Wilson formula) and an exponent η which differs only by a numerical constant $6C\ln L/(L^2-1)=0.35$ (for $C=\frac{1}{4}$ and $L=2$) compared with $\frac{1}{2}=0.5$.

Otherwise, we would like to follow numerically the non-trivial solution starting from $d=4$ up to $d\to 2$. In particular, because of the searching algorithm we are interested in the codimension of the fixed point. The problem may come from the coefficients r_j which have exponent

$$y_{r_j}^{-1} = d-j\alpha = d(1-j)+2j \tag{56}$$

and $y_{r_j}^{-1}<0$ for $d<d_c(j)=2+2/(j-1)$. At zero under in ε, the coefficients $r_3, r_4, r_5, \ldots$ become successively relevant for dimensions $3, 2+\frac{2}{3}, 2+\frac{1}{2}, \ldots$. Thus, we want to calculate at first order in ε the exponents $y_{r_j}^{-1}$ and, after an analysis in powers of ε, we see that the relevant contribution is given by the term $r_j\lambda$ corresponding to the diagram



The combinatorial factor which comes from the expansion of the exponential is $\frac{1}{2}\times 2=1$ and the contribution of this diagram is

$$2j\langle(\varphi^{2j-2}y^2+2j(j-1)\varphi^{2j-4}(\varphi y)^2)\cdot(\varphi^2y^2+2(\varphi y)^2\rangle_c = \varphi^{2j}j(n+6j-4).$$

We obtain the exponent of r_j at first order in ε

$$y_{r_j}^{-1} = d - j(d-2) + (d-4)j\frac{n+6j-4}{n+8}. \tag{57}$$

Finally, for all j, for $2 \leqslant d \leqslant 4$, then $y_{r_j}^{-1} < 0$ and all the variables r_j are irrelevant at first order in ε. Thus, the fixed point bifurcating at $d = 4$ should be of codimension one up to $d = 2$.

3.5. Numerical resolution of the Golner formulae

3.5.1. Explicit form of the recursion relations. To find numerical fixed points of equations (37) for the vectorial model (51), it is essential to use the rotational symmetry to obtain iteration formulae in which the number of components n of φ appears only as a parameter. We take R in $O(n)$ such that

$$\boldsymbol{\varphi} = \boldsymbol{\varphi} R \boldsymbol{e}_1 \qquad \boldsymbol{e}_1 = (1, 0, 0, \ldots) \tag{58}$$

with the notation $\varphi = \|\boldsymbol{\varphi}\|$. Thus we have

$$R_{\alpha 1} = \frac{\varphi_\alpha}{\varphi}. \tag{59}$$

We make the same rotation on y

$$\boldsymbol{y} = R\boldsymbol{y}' \qquad \boldsymbol{y}' = (y_\parallel, y_\perp \cdot \boldsymbol{e}_\perp) \qquad \boldsymbol{e}_1 \cdot \boldsymbol{e}_\perp = 0 \tag{60}$$

and obtain $\boldsymbol{y} \cdot \boldsymbol{\varphi} = \varphi \boldsymbol{y}' \cdot R^+ \cdot \boldsymbol{e}_1 = \varphi y_\parallel$. The integrals on y transform themselves into integrals on y' and by using the residual symmetry under $O(n-1)$ for $y_\perp$ and $O(1)$ for $y_\parallel$

$$\int_{\mathbb{R}^n} \mathrm{d}^n y = 2\Omega_{n-2} \int_0^\infty \mathrm{d}y_\perp \, y_\perp^{n-2} \int_0^\infty \mathrm{d}y_\parallel. \tag{61}$$

The + operator is

$$Q^+(\varphi, y) = \tfrac{1}{2} Q(\sqrt{\varphi^2 + y^2 + 2\varphi y_\parallel}) + \tfrac{1}{2} Q(\sqrt{\varphi^2 + y^2 - 2\varphi y_\parallel}) \tag{62}$$

and the measure which appears in the averages is

$$P = \boldsymbol{y} W(\varphi) \boldsymbol{y} + Q^+(\boldsymbol{\varphi}, \boldsymbol{y}) = y_\perp^2 f(\varphi) + y_\parallel^2 (f(\varphi) + \varphi^2 g(\varphi)) + Q^+(\varphi^2 + y^2, \varphi y_\parallel). \tag{63}$$

Now n is just a parameter and these formulae allow us to calculate $I(\varphi)$. According to the same idea, we have still to obtain an explicit form for W', i.e. for f' and g'. However, the calculations are quite long and the formulae cumbersome; therefore we do not give this piece of algebra. We just want to add that, in some terms, the limit $\varphi \to 0$ is of the kind 0/0 and has to be removed by using l'Hospital's rule.

3.5.2. Numerical problems. In order to solve numerically the Golner formulae, we have to overcome at least all the problems of the Wilson formula and some specific ones as well. In particular, the convergence problem is more difficult, because the space of Hamiltonians is bigger, i.e. there are three functions instead of one. Moreover, close to two dimensions, as we get closer to the fixed point, some oscillations appear with the interations of the kind $\|Q_n - Q_{n-1}\| \sim \cos(\omega n) \exp(-\gamma n)$, γ small. Compared

with the Wilson formula, the computational times increase dramatically because we need to calculate 17 integrals instead of one for each iteration, and the convergence is slower. Thus, it is essential for this problem to use an efficient algorithm. We use essentially the same method as developed for the Wilson formula, with the natural replacement of the 'norm' by

$$\|x(\varphi)\|^2 = \frac{\int \mathrm{d}\varphi \, \exp(-\varphi^2 f(\varphi) - \varphi^4 g(\varphi) - Q(\varphi)) x^2(\varphi)}{\int \mathrm{d}\varphi \, \exp(-\varphi^2 f(\varphi) - \varphi^4 g(\varphi) - Q(\varphi))}. \tag{64}$$

A specific problem of the Golner formulae is the numerical approximations in the computation of the fluctuations. More specifically, the three functions f, g and Q are known only on a given mesh. When we compute the new functions after one iteration of the renormalisation group, we obtain the new 'exact' functions, up to a numerical noise coming from the mesh and from the fact that the integrals are computed with a finite number of points. This is not important in the Wilson formula in which there is only one integral which will average this noise. By contrast, in the Golner formulae, averages of squares of derivatives in the computation of W' appear and, if we do not take precautions, we will obtain essentially the fluctuations of the numerical noise instead of the fluctuations of the functions. Thus, it is very important to smooth the functions, and this is done thanks to the procedure described in section 2.3.1. This method works correctly if the smoothing length is greater than the lattice spacing and at least in the range of variation of the functions.

Another difficult problem is to compute the new functions f and g at $\varphi = 0$ and for φ close to zero. This is important because the exponent η is given by the condition $f'(0) = 1$. As we mention above, the indeterminations of the singular terms $0/0$ at $\varphi = 0$ are removed with l'Hospital's rule. Thus, we need to write a specific integration subroutine for this point in which, moreover, we have to remove the singularities of the integrands at $y = 0$. Now, for small φ, we have to compute expressions of the kind $\langle h(\varphi)\rangle/\varphi$ with $\langle h(\varphi)\rangle \to 0$ by symmetry. Because the symmetries are not completely respected in the numerical computations, the function $\langle h(\varphi)\rangle$ does not go to zero, and the term $\langle h(\varphi)\rangle/\varphi$ becomes singular for φ small. Then the idea is to calculate $(\langle h(\varphi)\rangle - \langle h(0)\rangle)/\varphi$ in which $\langle h(0)\rangle$ means the numerical computation of $\langle h(\varphi)\rangle$ at $\varphi = 0$. We proceed in the same way for the $1/\varphi^2$ singularities, and we check that $f(\varphi)\,(g/\varphi))$ goes to $f(0)\,(g(0))$ for $\varphi \to 0$ at least with a small discrepancy.

Beyond the mesh, it is more difficult to extrapolate the functions f and g than the potential Q, which has a typical behaviour. We use the same extrapolating function as described in section 2.3.1 for Q, but with a rougher fit of the parameters. Finally, a part of the program needs to be adapted for the much simpler case $n = 1$.

3.5.3. Numerical result. The Golner formulae explicitly depend on the dimension as the Wilson formula and also through the two constants C and ρ. Nevertheless, for dimensions between 2 and 4, C and ρ depend weakly on the dimension, so we fix for the numerical calculations

$$C = \rho = \tfrac{1}{4}. \tag{65}$$

Below two dimensions, C and ρ differ noticeably from the value, and we have done a set of calculations for the one-component model close to two dimensions with the constants C and ρ calculated with the formulae (47) and (48).

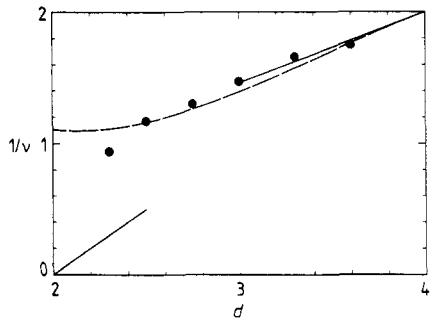


Figure 14. Critical exponent $1/\nu$ as a function of space dimension for the five-component model. The full curves represent the $2+\varepsilon$ and $4-\varepsilon$ expansions to first order; the broken curve represents the $4-\varepsilon$ expansion at order ε^4.

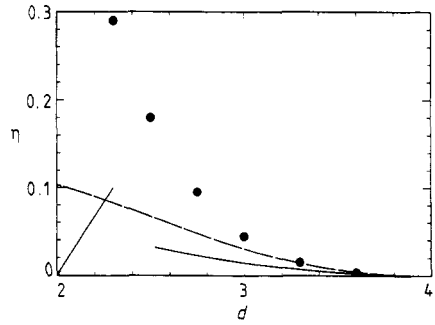


Figure 15. Critical exponent η as a function of space dimension for the five-component model. The full curves represent the $2+\varepsilon$ and $4-\varepsilon$ expansions to first order, and the broken curve gives the $4-\varepsilon$ expansion to order ε^4.

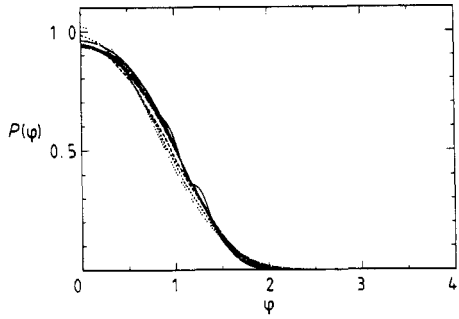


Figure 16. Probability P of φ at the fixed point for the one-component model. In order of increasing dash size we have the dimensions $d = 3.6$, 3.3, 3.0, 2.75, 2.5, 2.3, 2.0 and 1.7 (full curve).

In order to be able to compare the numerical results obtained with the Wilson and Golner formulae, we chose to make essentially the same measurements:

for $n = 5$	$2 \leqslant d \leqslant 4$	$1/\nu, \eta$
for $n = 1$	$1.7 \leqslant d \leqslant 4$	$P, 1/\nu, \eta$
for $d = 3$	$1 \leqslant n \leqslant 20$	$Q, f, g, P, 1/\nu, \eta$.

See figures 14–22.

It has not been possible to get fixed points close to two dimensions for the models with a continuous symmetry. The reason is that the functions f, g and Q developed singularities for $d \to 2$. By contrast, the model with discrete symmetry $n = 1$ stays perfectly smooth when the dimension decreases and it has been possible to obtain fixed point down to $d = 1.7$.

It is more difficult to run this program compared with the Wilson program because the computational times are much bigger and the convergence less robust. However, when we can compare the results, i.e. for Q, P and the exponent $1/\nu$, they show no significant differences, except for $n = 1$ close to two dimensions. We will now discuss in detail each set of data.

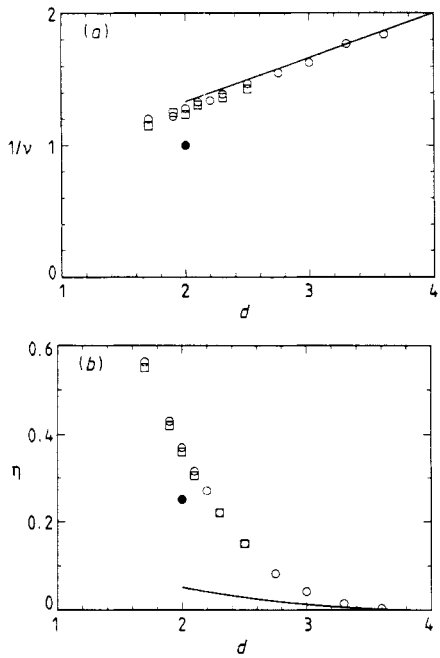


Figure 17. (*a*) Critical exponent $1/\nu$ as a function of spatial dimension for the one-component model. (*b*) Critical exponent η as a function of spatial dimension for the one-component model. The open circles represent the results of the calculations with $C = \rho = \frac{1}{4}$ and the open squares represent the results of the calculation with $C(\delta)$ and $\rho(d)$. The full curve denotes the $4-\varepsilon$ expansion to first order, and the full circle represents Onsager's result.

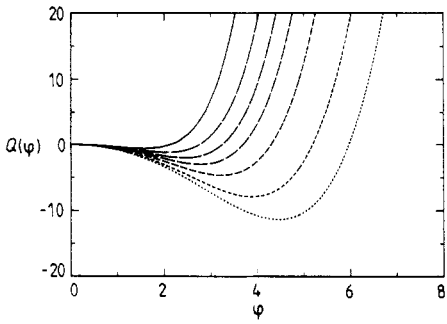


Figure 18. Potential $Q(\varphi)$ at the fixed point at three dimensions. In order of decreasing dash size, we have the models $n = 1$ (full curve), 3, 5, 7, 10, 15, 20.

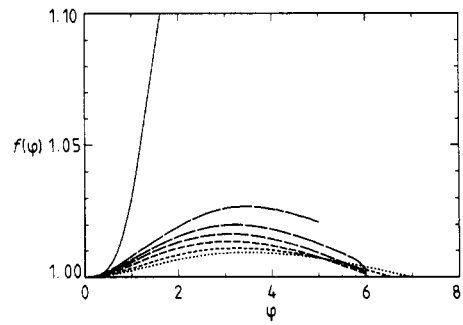


Figure 19. The function $f(\varphi)$ at the fixed point at three dimensions. In order of decreasing dash size, we have the models $n = 1$ (full curve), 3, 5, 7, 10, 15, 20.

The potential and $P(\varphi)$ show, as with the Wilson formula, how the five-component model goes continuously with the dimension from a φ^4-like model close to four dimensions to a nonlinear σ-model close to two dimensions. We also observe that the kinetic parts f and g become singular when $d \to 2$. These singularities come from the derivatives of the potential, which tends to be deeply peaked in order to approximate the nonlinear σ-model S_4. For the $1/\nu$ exponent, we are very close to the $4-\varepsilon$ expansion, but we do not see any tendency to recover the $2+\varepsilon$ expansion. For the η

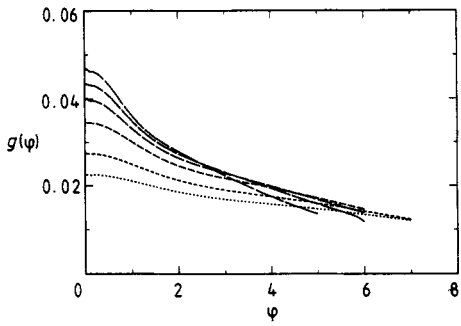


Figure 20. The function $g(\varphi)$ at the fixed point at three dimensions. In order of decreasing dash size, we have the models $n = 1$ (full curve), 3, 5, 7, 10, 15, 20.

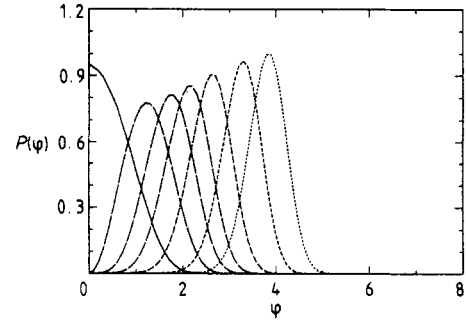


Figure 21. Probability P of φ at the fixed point at three dimensions. In order of decreasing dash size, we have the models $n = 1$ (full curve), 3, 5, 7, 10, 15, 20.

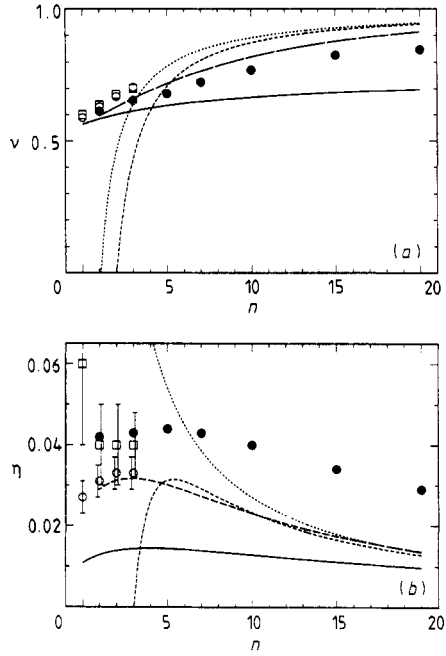


Figure 22. (*a*) Critical exponent ν at three dimensions as a function of the number of components of the model. (*b*) Critical exponent η at three dimensions as a function of the number of components of the model. The full circles denote the numerical calculations, the full (respectively, long-broken) curve denotes the ε expansion to first order (respectively order ε^4), the open circles indicate the λ expansion of Le Guillou and Zinn-Justin, the open squares represent the high-temperature expansion, and the dotted (respectively, short-broken) curve denotes the $1/n$ expansion at order $1/n$ (respectively, $1/n^2$ for ν and $1/n^3$ for η).

exponent, the $4-\varepsilon$ expansion at first order is out by a factor of 3 (see below), and η seems to diverge for $d \to 2$. Thus, the Golner formulae miss something when the potential and W become singular; this is quite normal because we used a Taylor expansion at first order to derive it.

For the one-component model, the surprising fact is the quasi-independence of $P(\varphi)$ from the dimension, despite the fact that Q and f vary strongly. Compared with the Wilson formula, the introduction of derivatives in the formulae completely smoothes the potential and $P(\varphi)$ when $d \to 2$. The fact that, up to two dimensions, the potential is φ^4-like can explain the success of the $4-\varepsilon$ expansion as far as $\varepsilon = 2$. However, for $d = 1.7$, we observe the appearance of oscillations, as with the Wilson formula for $d \leqslant 2.3$, indicating that the potential becomes singular at lower dimension (it has not been possible to obtain the convergence for $d \leqslant 1.5$). The exponent $1/\nu$ behaves as expected and agrees well with Onsager's result at two dimensions. For the η exponent, we have similar results as for the five-component model (but here the lower critical dimension is 1). The calculations made with $C(d)$ and $\rho(d)$ compared with $C = \rho = \frac{1}{4}$, show the very weak dependences of the results on the value of these constants, despite what the $4-\varepsilon$ expansion can suggest for the η exponent.

From this analysis regarding the dimension for a model with continuous symmetry and a model with discrete symmetry, we will keep in mind that the Golner formulae give very good results when the Hamiltonian is sufficiently smooth.

For Q, P and ν, the results at three dimensions for various models give almost identical results compared with the Wilson formula. For $n = 1$, the function g is absorbed in f, which explains the discrepancy for f between $n = 1$ and $n > 1$. The ν exponent is very close to the $4-\varepsilon$ expansion, but the η exponent is three times bigger compared to the same expansion. However, at three dimensions, for $n = 1$, 2 and 3, we dispose of higher-order perturbation expansions, i.e. high-temperature expansion, $4-\varepsilon$ expansion Borel transformed [15], and expansion in the coupling constant λ also Borel transformed. This last method is presented in an article by Le Guillou and Zinn-Justin [16]; this article includes also a table of the results obtained by the other methods, as well as experimental data. We also have at our disposal numerical Monte Carlo RG on the Ising model [17], and $1/n$ expansion at order $1/n^3$ [18]. The main results are summed up in the table below and in the graph for ν and η. For the most studied case, the Ising and the Heisenberg model, our results are summarised in Table 1.

Table 1. Comparison of different analyses of the $n = 1$ and $n = 3$ models.

	$n = 1$		$n = 3$	
	ν	η	ν	η
Our numerical results	0.61	0.042	0.654	0.043
High-temperature	0.638 ± 0.002	0.041 ± 0.01	0.70 ± 0.02	0.040 ± 0.008
Expansion in λ	0.630 ± 0.0015	0.031 ± 0.004	0.705 ± 0.0030	0.033 ± 0.004
Expansion in ε [15]	0.6305 ± 0.0025	0.037 ± 0.003	0.710 ± 0.007	0.040 ± 0.003
Ising [17]	0.629 ± 0.004	0.031 ± 0.005		

The agreement is very impressive. For the ν exponent, all the methods give similar results, although ours slightly underestimate ν. By contrast, the η exponent is very difficult to evaluate analytically but the program converges rapidly to a stable value. This exponent is certainly a fine test of a calculation. It shows that the ε expansion at first order is not sufficient, but all the other methods give results inside the error bars.

On the other hand, the result for η seems to deteriorate for large n, if we compare it with the ϵ expansion to fourth order [14], or to the $1/n$ expansion to third order [18]. But for the exponent ν the difference is not so large.

As a conclusion for these numerical calculations with the Golner formulae, we keep in mind the very good agreement for the critical exponents compared with the calculations made with sophisticated methods of field theory. This shows that the Golner formulae include the essential part of the physics when the Hamiltonian is sufficiently smooth.

Acknowledgment

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